A FINITENESS THEOREM FOR ALGEBRAIC CYCLES

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ABSTRACT. Let X be a smooth projective variety. Starting with a finite set of cycles on powers X^m of X, we consider the $\mathbf{Q}$-vector subspaces of the $\mathbf{Q}$-linear Chow groups of the X^m obtained by iterating the algebraic operations and pullback and push forward along those morphisms $X^l \to X^m$ for which each component $X^l \to X$ is a projection. It is shown that these $\mathbf{Q}$-vector subspaces are all finite-dimensional, provided that the $\mathbf{Q}$-linear Chow motive of X is a direct summand of that of an abelian variety.

1. INTRODUCTION

Let X be a smooth projective variety over a field F. Starting with a finite set of cycles on powers X^m of X, consider the $\mathbf{Q}$-vector subspaces C_m of the $\mathbf{Q}$-linear Chow groups $CH(X^m)_{\mathbf{Q}}$ formed by iterating the algebraic operations and pullback p^* and push forward p_* along those morphisms $p : X^l \to X^m$ for which each component $X^l \to X$ is a projection. It is plausible that the C_m are always finite-dimensional, because when F is finitely generated over the prime field it is plausible that the $CH(X^m)_{\mathbf{Q}}$ themselves are finite-dimensional. In this paper we prove the finite-dimensionality of the C_m when the $\mathbf{Q}$-linear Chow motive of X is a direct summand of that of an abelian variety over F.

More precisely, suppose given an adequate equivalence relation $\sim$ on $\mathbf{Q}$-linear cycles on smooth projective varieties over F. We say that X is a *Kimura variety for* $\sim$ if, in the category of $\mathbf{Q}$-linear Chow motives modulo $\sim$, the motive of X is the direct sum of one for which some exterior power is 0 and one for which some symmetric power is 0. A Kimura variety for $\sim$ is also a Kimura variety for any coarser equivalence relation. It is known (e.g. [4], Corollary 4.4) that if the $\mathbf{Q}$-linear Chow motive of X is a direct summand of that of an abelian variety, then X is a Kimura variety for any $\sim$. The main result is now Theorem 1.1 below. In addition to a finiteness result, it contains also a nilpotence result. By a filtration $C^\bullet$ on a graded $\mathbf{Q}$-algebra C we mean a descending sequence $C = C^0 \supset C^1 \supset \ldots$ of graded ideals of C such that $C^r.C^s \subset C^{r+s}$ for every r and s. The morphisms $p : X^l \to X^m$ in Theorem 1.1 (a) are exactly those defined by maps $[1,m] \to [1,l]$.

Theorem 1.1. *Let X be a smooth projective variety over F which is a Kimura variety for the equivalence relation $\sim$. For $n = 0, 1, 2, \ldots,$ let Z_n be a finite subset of $CH(X^n)_{\mathbf{Q}}/\sim$, with Z_n empty for n large. Then there exists for each n a graded $\mathbf{Q}$-subalgebra C_n of $CH(X^n)_{\mathbf{Q}}/\sim$, and a filtration $(C_n)^\bullet$ on C_n, with the following properties.*

(a) *If $p : X^l \to X^m$ is a morphism for which each component $X^l \to X$ is a projection, then p^* sends C_m to C_l and p_* sends C_l to C_m, and p^* and p_* respect the filtrations on C_l and C_m.*

(b) *For every n, the $\mathbf{Q}$-algebra C_n is finite-dimensional and contains Z_n.*

(c) *For every n, the cycles in C_n which are numerically equivalent to 0 lie in $(C_n)^1$, and we have $(C_n)^r = 0$ for r large.*

The finiteness result of Theorem 1.1 is non-trivial only if Z_n is non-empty for some $n > 1$. Indeed it will follow from Proposition 2.1 below that for any smooth projective variety X over F and finite subset Z_1 of $CH(X)_{\mathbf{Q}}$, there is a finite-dimensional graded $\mathbf{Q}$-subalgebra C_n of $CH(X^n)_{\mathbf{Q}}$ for each n such that C_1 contains Z_1 and p^* sends C_m to C_l and p_* sends C_l to C_m for every p as in (a) of Theorem 1.1.

If X is a Kimura variety for $\sim$, then the ideal of correspondences numerically equivalent to 0 in the algebra $CH(X \times_F X)_{\mathbf{Q}}/\sim$ of self-correspondences of X has been shown by Kimura ([4], Proposition 7.5) to consist of nilpotent elements, and by André and Kahn ([1], Proposition 9.1.14) to be in fact nilpotent. The nilpotence result of Theorem 1.1 implies that of Kimura, but neither implies nor is implied by that of André and Kahn.

If $\sim$ is numerical equivalence, then the $CH(X^m)_{\mathbf{Q}}/\sim = \overline{CH}(X^m)_{\mathbf{Q}}$ are all finite dimensional. The following result shows that they are generated in a suitable sense by the $\overline{CH}(X^m)_{\mathbf{Q}}$ for m not exceeding some fixed n.

Theorem 1.2. *Let X be a smooth projective variety over F which is a Kimura variety for numerical equivalence. Then there exists an integer $n \geq 0$ with the following property: for every m, the $\mathbf{Q}$-vector space $\overline{CH}(X^m)_{\mathbf{Q}}$ is generated by elements of the form*

$$(1.1) \qquad\qquad p_*((p_1)^*(z_1).(p_2)^*(z_2).\cdots.(p_r)^*(z_r)),$$

where z_i lies in $\overline{CH}(X^{m_i})_{\mathbf{Q}}$ with $m_i \leq n$, and $p : X^l \to X^m$ and the $p_i : X^l \to X^{m_i}$ are morphisms for which each component $X^l \to X$ is a projection.

Theorem 1.1 will be proved in Section 7 and Theorem 1.2 in Section 8. Both theorems are deduced from the following fact. Given a Kimura variety X for $\sim$, there is a reductive group G over $\mathbf{Q}$, a finite-dimensional G-module E, a central $\mathbf{Q}$-point ρ of G with $\rho^2 = 1$, and a commutative algebra R in the tensor category $\mathrm{REP}(G, \rho)$ of G-modules with symmetry modified by ρ, with the following property. If we write $\mathcal{M}_\sim(F)$ for the category of ungraded motives over F modulo $\sim$ and E_R for the free R-module $R \otimes_k E$ on E, then there exist isomorphisms

$$\xi_{r,s} : \mathrm{Hom}_{G,R}((E_R)^{\otimes r}, (E_R)^{\otimes s}) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{M}_\sim(F)}(h(X)^{\otimes r}, h(X)^{\otimes s})$$

which are compatible with composition and tensor products of morphisms and with symmetries interchanging the factors E_R and $h(X)$. These isomorphisms arise because there is a fully faithful tensor functor from the category of finitely generated free R-modules to $\mathcal{M}_\sim(F)$, which sends E_R to $h(X)$ (see [7], Lemma 3.3 for a similar result). However to keep the exposition as brief as possible, the $\xi_{r,s}$ will simply be constructed directly here, in Sections 5 and 6. Now we have an equality

$$CH(X^r)_{\mathbf{Q}}/\sim = \mathrm{Hom}_{\mathcal{M}_\sim(F)}(\mathbf{1}, h(X)^{\otimes r})$$

of $\mathbf{Q}$-algebras, and pullback along a morphism $f : X^l \to X^m$ is given by composition with $h(f)$. There is a canonical autoduality $h(X)^{\otimes 2} \to \mathbf{1}$ on $h(X)$, and push

forward along f is given by composing with the transpose of $h(f)$ defined by this autoduality. Using the isomorphisms $\xi_{r,s}$, Theorems 1.1 and 1.2 then reduce to easily solved problems about the G-algebra R.

2. GENERATED SPACES OF CYCLES

The following result gives an explicit description of the spaces of cycles generated by a given set of cycles on the powers of an arbitrary smooth projective variety X. By the top Chern class of X we mean the element $(\Delta_X)^*(\Delta_X)_*(1)$ of $CH(X)_\mathbf{Q}$, where $\Delta_X : X \to X^2$ is the diagonal. We define the tensor product $z \otimes z'$ of z in $CH(X)_\mathbf{Q}$ and z' in $CH(X')_\mathbf{Q}$ as $\mathrm{pr}_1{}^*(z).\mathrm{pr}_2{}^*(z')$ in $CH(X \times_F X')_\mathbf{Q}$. The push forward of $z \otimes z'$ along a morphism $f \times f'$ is then $f_*(z) \otimes f'_*(z')$.

Proposition 2.1. *Let X be a smooth projective variety over F. For $m = 0, 1, 2, \ldots$ let Z_m be a subset of $CH(X^m)_\mathbf{Q}$, such that Z_1 contains the top Chern class of X. Denote by C_m the $\mathbf{Q}$-vector subspace of $CH(X^m)_\mathbf{Q}$ generated by elements of the form*

$$(2.1) \qquad p_*((p_1)^*(z_1).(p_2)^*(z_2).\cdots.(p_n)^*(z_r)),$$

where z_i lies in Z_{m_i}, and $p : X^j \to X^m$ and the $p_i : X^j \to X^{m_i}$ are morphisms for which each component $X^j \to X$ is a projection. Then C_m is a $\mathbf{Q}$-subalgebra of $CH(X^m)_\mathbf{Q}$ for each m. If $q : X^l \to X^m$ is a morphism for which each component $X^l \to X$ is a projection, then q^* sends C_m into C_l and q_* sends C_l into C_m.

Proof. Write $\mathcal{P}_{l,m}$ for the set of morphisms $X^l \to X^m$ for which each component $X^l \to X$ is a projection. Then the composite of an element of $\mathcal{P}_{j,l}$ with an element of $\mathcal{P}_{l,m}$ lies in $\mathcal{P}_{j,m}$. Thus $q_*(C_l) \subset C_m$ for q in $\mathcal{P}_{l,m}$. Similarly the product of two elements of $\mathcal{P}_{j,m}$ lies in $\mathcal{P}_{2j,2m}$. Thus the tensor product of two elements of C_m lies in C_{2m}. Since the product of two elements of $CH(X^m)_\mathbf{Q}$ is the pullback of their tensor product along $\Delta_{X^m} \in \mathcal{P}_{m,2m}$, it remains only to show that

$$(2.2) \qquad q^*(C_m) \subset C_l$$

for q in $\mathcal{P}_{l,m}$. This is clear when $l = m$ and q is a symmetry σ permuting the factors X of X^m, because $\sigma^* = (\sigma^{-1})_*$. An arbitrary q factors for some l' as $q'' \circ q'$ with q' in $\mathcal{P}_{l,l'}$ a projection and q'' in $\mathcal{P}_{l',m}$ a closed immersion. It thus is enough to show that (2.2) holds when q is a projection or q is a closed immersion.

Suppose that q is a projection. If we write

$$w_{s,n} : X^{s+n} \to X^n$$

for the projection onto the last n factors, then q is a composite of a symmetry with $w_{l-m,m}$. Thus (2.2) holds because $w_{l-m,m}{}^* = 1 \otimes -$ sends C_m into C_l.

Suppose that q is a closed immersion. Then q is a composite of the

$$e_s = X^{s-2} \times \Delta_X : X^{s-1} \to X^s$$

for $s \geq 2$ and symmetries. To prove (2.2), we may thus suppose that $m \geq 2$ and $q = e_m$.

Denote by W_m the $\mathbf{Q}$-subalgebra of $CH(X^m)_\mathbf{Q}$ generated by the $v^*(Z_{m'})$ for any m' and v in $\mathcal{P}_{m,m'}$. Then u^* sends W_m into W_l for any u in $\mathcal{P}_{l,m}$, and by the projection formula C_m is a W_m-submodule of $CH(X^m)_\mathbf{Q}$. Since C_m is the sum of the $p_*(W_j)$ with p in $\mathcal{P}_{j,m}$, it is to be shown that

$$(2.3) \qquad (e_m)^* p_*(W_j) \subset C_{m-1}$$

for every p in $\mathcal{P}_{j,m}$. We have $p = w_{j,m} \circ \Gamma_p$ with Γ_p the graph of p, and

$$(e_m)^* \circ (w_{j,m})_* = (w_{j,m-1})_* \circ (e_{j+m})^*.$$

Thus (2.3) will hold provided that $(e_{j+m})^*(\Gamma_p)_*(W_j) \subset C_{j+m-1}$. Replacing m by $j + m$ and p by Γ_p, we may thus suppose that p has a left inverse in $\mathcal{P}_{m,j}$. In that case any y in W_j is of the form $p^*(x)$ with x in W_m, and then

$$(e_m)^* p_*(y) = (e_m)^*(p_*(1).x) = (e_m)^* p_*(1).(e_m)^*(x).$$

Thus (2.3) will hold provided that $(e_m)^* p_*(1)$ lies in C_{m-1}.

To see that $(e_m)^* p_*(1)$ lies in C_{m-1}, note that e_m has a left inverse f in $\mathcal{P}_{m,m-1}$. Then

$$(e_m)^* p_*(1) = f_*(e_m)_*(e_m)^* p_*(1) = f_*(p_*(1).(e_m)_*(1)) = f_* p_* p^*(e_m)_*(1).$$

Since $(e_m)_*(1) = (w_{m-2,2})^*(\Delta_X)_*(1)$, we reduce finally to showing that $h^*(\Delta_X)_*(1)$ lies in C_j for every h in $\mathcal{P}_{j,2}$. Such an h factors as a projection followed by either a symmetry of X^2 or Δ_X, so we may suppose that $j = 1$ and $h = \Delta_X$. Then $h^*(\Delta_X)_*(1)$ is the top Chern class of X, which by hypothesis lies in $Z_1 \subset C_1$. $\square$

3. Group representations

Let k be a field. By a k-linear category we mean a category equipped with a structure of k-vector space on each hom-set such that the composition is k-bilinear. A k-linear category is said to be pseudo-abelian if it has a zero object and direct sums, and if every idempotent endomorphism has an image. A *k-tensor category* is a pseudo-abelian k-linear category $\mathcal{C}$, together with a structure of symmetric monoidal category on $\mathcal{C}$ ([5], VII 7) such that the tensor product $\otimes$ is k-bilinear on hom-spaces. Thus $\mathcal{C}$ is equipped with a unit object $\mathbf{1}$, and natural isomorphisms

$$(L \otimes M) \otimes N \xrightarrow{\sim} L \otimes (M \otimes N),$$

the associativities,

$$M \otimes N \xrightarrow{\sim} N \otimes M,$$

the symmetries, and $\mathbf{1} \otimes M \xrightarrow{\sim} M$ and $M \otimes \mathbf{1} \xrightarrow{\sim} M$, which satisfy appropriate compatibilities. We assume in what follows that $\mathbf{1} \otimes M \xrightarrow{\sim} M$ and $M \otimes \mathbf{1} \xrightarrow{\sim} M$ are identities: this can be always arranged by replacing if necessary $\otimes$ by an isomorphic functor. Brackets in multiple tensor products will often be omitted when it is of no importance which bracketing is chosen. The tensor product of r copies of M will then be written as $M^{\otimes r}$, and similarly for morphisms. There is a canonical action $\tau \mapsto M^{\otimes \tau}$ of the symmetric group $\mathfrak{S}_r$ on $M^{\otimes r}$, defined using the symmetries. It extends to a homomorphism of k-algebras from $k[\mathfrak{S}_r]$ to $\mathrm{End}(M^{\otimes r})$. When k is of characteristic 0, the symmetrising idempotent in $k[\mathfrak{S}_r]$ maps to an idempotent endomorphism e of $M^{\otimes r}$, and we define the rth symmetric power $S^r M$ of M as the image of e. Similarly we define the rth exterior power $\bigwedge^r M$ of M using the antisymmetrising idempotent in $k[\mathfrak{S}_r]$

Let G be a linear algebraic group over k. We write $\mathrm{REP}(G)$ for the category of G-modules. The tensor product $\otimes_k$ over k defines on $\mathrm{REP}(G)$ a structure of k-tensor category. Recall ([9], 3.3) that every G-module is the filtered colimit of its finite-dimensional G-submodules. If E is a finite-dimensional G-module, then regarding $\mathrm{REP}(G)$ as a category of comodules ([9], 3.2) shows that $\mathrm{Hom}_G(E, -)$ preserves filtered colimits. When k is algebraically closed, a k-vector subspace of a

G-module is a G-submodule provided it is stable under every k-point of G. This is easily seen by reducing to the finite-dimensional case.

We suppose from now on that k has characteristic 0. Let ρ be a central k-point of G with $\rho^2 = 1$. Then ρ induces a $\mathbf{Z}/2$-grading on $\mathrm{REP}(G)$, with the G-modules pure of degree i those on which ρ acts as $(-1)^i$. We define as follows a k-tensor category $\mathrm{REP}(G, \rho)$. The underlying k-linear category, tensor product and associativities of $\mathrm{REP}(G, \rho)$ are the same as those of $\mathrm{REP}(G)$, but the symmetry $M \otimes N \xrightarrow{\sim} N \otimes M$ is given by multiplying that in $\mathrm{REP}(G)$ by $(-1)^{ij}$ when M is of degree i and N of degree j, and then extending by linearity. When $\rho = 1$, the k-tensor categories $\mathrm{REP}(G)$ and $\mathrm{REP}(G, \rho)$ coincide.

An algebra in a k-tensor category is defined as an object R together with a multiplication $R \otimes R \to R$ and unit $\mathbf{1} \to R$ satisfying the usual associativity and identity conditions. Since the symmetry is not used in this definition, an algebra in $\mathrm{REP}(G, \rho)$ is the same as an algebra in $\mathrm{REP}(G)$, or equivalently a G-algebra. An algebra R in $\mathrm{REP}(G, \rho)$ will be said to be finitely generated if its underlying k-algebra is. It is equivalent to require that R be generated as a k-algebra by a finite-dimensional G-submodule.

A (left) module over an algebra R is an object N equipped with an action $R \otimes N \to N$ satisfying the usual associativity and identity conditions. If R is an algebra in $\mathrm{REP}(G, \rho)$ or $\mathrm{REP}(G)$, we also speak of a (G, R)-module. A (G, R)-module is said to be finitely generated if it is finitely generated as a module over the underlying k-algebra of R. It is equivalent to require that it be generated as a module over the k-algebra R by a finite-dimensional G-submodule.

An algebra R in a k-tensor category is said to be commutative if composition with the symmetry interchanging the factors R in $R \otimes R$ leaves the multiplication unchanged. If R is an algebra in $\mathrm{REP}(G, \rho)$, this notion of commutativity does not in general coincide with that of the underlying k-algebra, but it does when ρ acts as 1 on R.

Coproducts exist in the category of commutative algebras in a k-tensor category: the coproduct of R and R' is $R \otimes R'$ with multiplication the tensor product of the multiplications of R and R' composed with the appropriate symmetry. To any map $[1, m] \to [1, l]$ and commutative algebra R there is then associated a morphism $R^{\otimes m} \to R^{\otimes l}$, defined using symmetries $R^{\otimes \tau}$ and the unit and multiplication of R and their tensor products and composites, such that each component $R \to R^{\otimes l}$ is the embedding into one of the factors.

Let R be a commutative algebra in $\mathrm{REP}(G, \rho)$. Then the symmetry in $\mathrm{REP}(G, \rho)$ defines on any R-module a canonical structure of (R, R)-bimodule. The category of (G, R)-modules has a structure of k-tensor category, with the tensor product $N \otimes_R N'$ of N and N' defined in the usual way as the coequaliser of the two morphisms

$$N \otimes_k R \otimes_k N' \to N \otimes_k N'$$

given by the actions of R on N and N', and the tensor product $f \otimes_R f'$ of $f : M \to N$ and $f' : M' \to N'$ as the unique morphism rendering the square

$$
\begin{array}{ccc}
M \otimes_R M' & \xrightarrow{f \otimes_R f'} & N \otimes_R N' \\
\uparrow & & \uparrow \\
M \otimes_k M' & \xrightarrow{f \otimes_k f'} & N \otimes_k N'
\end{array}
$$

commutative.

Let P be an object in $\mathrm{REP}(G, \rho)$. We write P_R for the object $R \otimes_k P$ in the k-tensor category of (G, R)-modules. A morphism of commutative algebras $R' \to R$ in $\mathrm{REP}(G, \rho)$ induces by tensoring with P a morphism of R'-modules $P_{R'} \to P_R$. For each l and m, extension of scalars along $R' \to R$ then gives a k-linear map

$$(3.1) \qquad \mathrm{Hom}_{G,R'}((P_{R'})^{\otimes m}, (P_{R'})^{\otimes l}) \to \mathrm{Hom}_{G,R}((P_R)^{\otimes m}, (P_R)^{\otimes l})$$

Explicitly, (3.1) sends f' to the unique morphism of (G, R)-modules f that renders the square

$$
\begin{array}{ccc}
(P_R)^{\otimes m} & \xrightarrow{\ f\ } & (P_R)^{\otimes l} \\
\uparrow & & \uparrow \\
(P_{R'})^{\otimes m} & \xrightarrow{\ f'\ } & (P_{R'})^{\otimes l}
\end{array}
$$

commutative, where the vertical arrows are those defined by $P_{R'} \to P_R$. If P is finite-dimensional, then for given commutative algebra R and f, there is a finitely generated G-subalgebra R' of R such that f is in the image of (3.1). This can be seen by writing R as the filtered colimit $\mathrm{colim}_\lambda R_\lambda$ of its finitely generated G-subalgebras, and noting that since $P^{\otimes m}$ is finite-dimensional, the composite of $P^{\otimes m} \to (P_R)^{\otimes m}$ with f factors through some $(P_{R_\lambda})^{\otimes l}$.

Suppose that G is reductive, or equivalently that $\mathrm{Hom}_G(P, -)$ is exact for every G-module P. Then $\mathrm{Hom}_G(P, -)$ preserves colimits for P finite-dimensional. In particular $(-)^G = \mathrm{Hom}_G(k, -)$ preserves colimits. If R is a commutative algebra in $\mathrm{REP}(G, \rho)$ with $R^G = k$, then R has a unique maximal G-ideal. Indeed $J^G = 0$ for $J \neq R$ a G-ideal of R, while $(J_1)^G = 0$ and $(J_2)^G = 0$ implies $(J_1 + J_2)^G = 0$.

Lemma 3.1. *Let G be a reductive group over a field k of characteristic 0 and ρ be a central k-point of G with $\rho^2 = 1$. Let R be a finitely generated commutative algebra in $\mathrm{REP}(G, \rho)$ with $R^G = k$, and N be a finitely generated R-module.*

  (i) *The k-vector space N^G is finite-dimensional.*
  (ii) *For every G-ideal $J \neq R$ of R, we have $(J^r N)^G = 0$ for r large.*

Proof. Every object P of $\mathrm{REP}(G, \rho)$ decomposes as $P_0 \oplus P_1$ where ρ acts as $(-1)^i$ on P_i. In particular $R = R_0 \oplus R_1$ with R_0 a G-subalgebra of R. Suppose that R is generated as an algebra by the finite-dimensional G-submodule M. Then R_0 is generated as an algebra by $M_0 + M_1^2$, and hence is finitely generated. Since R is a commutative algebra in $\mathrm{REP}(G, \rho)$, it is generated as an R_0-module by M_1. Hence any finitely generated R-module is finitely generated as an R_0-module.

To prove (i), we reduce after replacing R by R_0 to the case where $R = R_0$. Then R is a commutative G-algebra in the usual sense. In this case it is well known that N^G is finite-dimensional over $k = R^G$ (e.g [8], II Theorem 3.25).

To prove (ii), note that $J_0 \neq R_0$ is an ideal of R_0. Since R is a finitely generated R_0-module, so also is R_1. If $x_1, x_2, \ldots, x_s$ generate R_1 over R_0, then since each x_i has square 0 we have $R_1^r = 0$ and hence $J_1^r = 0$ for $r > s$. Thus for $r > s$ we have

$$J^r N = (J_0 + J_1)^r N = J_0^r N + J_0^{r-1} J_1 N + \cdots + J_0^{r-s} J_1^s N \subset J_0^{r-s} N.$$

Replacing R by R_0 and J by J_0, we thus reduce again to the case where $R = R_0$ is a commutative G-algebra in the usual sense. We may suppose further that k is algebraically closed.

By (i), it is enough to show that $\bigcap_{r=0}^{\infty} J^r N = 0$, or equivalently ([2] III §3 No. 2 Proposition 5 and IV §1 No. 1 Proposition 2, Corollaire 2) that $J + \mathfrak{p} \neq R$ for every associated prime $\mathfrak{p}$ of N. Fix such a $\mathfrak{p}$, and consider the intersection $\mathfrak{p}'$ of the $g\mathfrak{p}$ for g in $G(k)$. It is stable under $G(k)$, and hence since k is algebraically closed is a G-ideal of R. Thus $J + \mathfrak{p}' \neq R$, because J and $\mathfrak{p}'$ are contained in the unique maximal G-ideal of R. Since each $g\mathfrak{p}$ lies in the finite set of associated primes of N, it follows that $J + g\mathfrak{p} \neq R$ for some g in $G(k)$. Applying g^{-1} then shows that $J + \mathfrak{p} \neq R$. $\qquad\square$

Let l_0 and l_1 be integers ≥ 0. Write

$$(3.2) \qquad\qquad G = GL_{l_0} \times_k GL_{l_1},$$

E_i for the standard representation of GL_{l_i}, regarded as a G-module, and

$$(3.3) \qquad\qquad E = E_0 \oplus E_1.$$

We may identify the endomorphism of E that sends E_i to itself and acts on it as $(-1)^i$ with a central k-point ρ of G with $\rho^2 = 1$.

Consider the semidirect product

$$(3.4) \qquad\qquad \Gamma_r = (\mathbf{Z}/2)^r \rtimes \mathfrak{S}_r,$$

where the symmetric group $\mathfrak{S}_r$ acts on $(\mathbf{Z}/2)^r$ through its action on $[1, r]$. For each r, the group Γ_r acts on $E^{\otimes r}$, with the action of $(\mathbf{Z}/2)^r$ the tensor product of the actions $i \mapsto \rho^i$ of $\mathbf{Z}/2$ on E, and the action of $\mathfrak{S}_r$ that defined by the symmetries in $\mathrm{REP}(G, \rho)$. Thus we obtain a homomorphism

$$(3.5) \qquad\qquad k[\Gamma_r] \to \mathrm{End}_G(E^{\otimes r})$$

of k-algebras.

For $r \leq r'$ we may regard Γ_r as a subgroup of $\Gamma_{r'}$, and hence $k[\Gamma_r]$ as a k-sub-algebra of $k[\Gamma_{r'}]$, by identifying $(\mathbf{Z}/2)^r$ with the subgroup of $(\mathbf{Z}/2)^{r'}$ with the last $r' - r$ factors the identity and $\mathfrak{S}_r$ with the subgroup of $\mathfrak{S}_{r'}$ which leaves the last $r' - r$ elements of $[1, r']$ fixed. Write e_0 for the idempotent of $k[\mathbf{Z}/2]$ given by half the sum of the two elements of $\mathbf{Z}/2$, and e_1 for $1 - e_0$. Given $\pi = (\pi_1, \pi_2, \ldots, \pi_r)$ in $(\mathbf{Z}/2)^r$, we then have an idempotent

$$e_\pi = e_{\pi_1} \otimes e_{\pi_2} \otimes \cdots \otimes e_{\pi_r}$$

in $k[\mathbf{Z}/2]^{\otimes r} = k[(\mathbf{Z}/2)^r] \subset k[\Gamma_r]$. When every component of π is $i \in \mathbf{Z}/2$, we write $e_{i,r}$ for e_π. We also write $a_{0,r}$ for the antisymmetrising idempotent and $a_{1,r}$ for the symmetrising idempotent in $k[\mathfrak{S}_r]$, and for $i \in \mathbf{Z}/2$ we write

$$(3.6) \quad x_{i,r} = e_{i,l_i+1} a_{i,l_i+1} = e_{i,l_i+1} a_{i,l_i+1} e_{i,l_i+1} = a_{i,l_i+1} e_{i,l_i+1} \in k[\Gamma_{l_i+1}] \subset k[\Gamma_r]$$

if $r > l_i$ and $x_{i,r} = 0$ otherwise.

Lemma 3.2. (i) *If $r \neq r'$ then $\mathrm{Hom}_G(E^{\otimes r}, E^{\otimes r'}) = 0$.*

(ii) *The homomorphism (3.5) is surjective, with kernel the ideal of $k[\Gamma_r]$ generated by $x_{0,r}$ and $x_{1,r}$.*

Proof. (i) The action of G on E restricts along the appropriate $\mathbf{G}_m \to G$ to the homothetic action of $\mathbf{G}_m$ on E.

(ii) Write I for the ideal of $k[\Gamma_r]$ generated by $x_{0,r}$ and $x_{1,r}$. The image of e_π under (3.5) is the projection onto the direct summand

$$E_\pi = E_{\pi_1} \otimes_k E_{\pi_2} \otimes_k \cdots \otimes_k E_{\pi_r}$$

of $E^{\otimes r}$. The e_π give a decomposition of the identity of $k[\Gamma_r]$ into orthogonal idempotents, and (3.5) is the direct sum over π and π' of the homomorphisms

$$(3.7) \qquad\qquad e_{\pi'} k[\Gamma_r] e_\pi \to \mathrm{Hom}_G(E_\pi, E_{\pi'})$$

it induces on direct summands of $k[\Gamma_r]$ and $\mathrm{End}_G(E^{\otimes r})$. It is thus enough to show that (3.7) is surjective, with kernel $e_{\pi'} I e_\pi$.

Restricting to the centre of G shows that the target of (3.7) is 0 unless π' and π have the same number of components 0 or 1, or equivalently $\pi' = \tau\pi\tau^{-1}$ for some $\tau \in \mathfrak{S}_r$. The same holds for the source of (3.7), because

$$\tau e_\pi \tau^{-1} = e_{\tau\pi\tau^{-1}}$$

for every τ and π. Since further the image of $\tau \in \mathfrak{S}_r$ under (3.5) induces an isomorphism from E_π to $E_{\tau\pi\tau^{-1}}$, to show that (3.7) has the required properties we may suppose that $\pi' = \pi$ and that $r = r_0 + r_1$ where the first r_0 components of π are 0 and the last r_1 are 1. Then the source of (3.7) has a basis $e_\pi \tau e_\pi = e_\pi \tau$ with τ in the subgroup $\mathfrak{S}_{r_0} \times \mathfrak{S}_{r_1}$ of $\mathfrak{S}_r$ that permutes the first r_0 and last r_1 elements of $[1, r]$ among themselves. Thus we may identify

$$k[\mathfrak{S}_{r_0}] \otimes_k k[\mathfrak{S}_{r_1}] = k[\mathfrak{S}_{r_0} \times \mathfrak{S}_{r_1}]$$

with the (non-unitary) k-subalgebra $e_\pi k[\Gamma_r] e_\pi$ of $k[\Gamma_r]$. Similarly we may identify

$$\mathrm{End}_G(E_0^{\otimes r_0}) \otimes_k \mathrm{End}_G(E_1^{\otimes r_1}) = \mathrm{End}_G(E_0^{\otimes r_0} \otimes_k E_1^{\otimes r_1})$$

with the (non-unitary) k-subalgebra $\mathrm{End}_G(E_\pi)$ of $\mathrm{End}_G(E)$. Now given τ and τ' in $\mathfrak{S}_r$, the element $e_\pi \tau' x_{i,r} \tau^{-1} e_\pi$ is 0 unless both τ and τ' send $[1, l_i + 1]$ into $[1, r_0]$ if $i = 0$ or into $[r_0 + 1, r]$ if $i = 1$. With the above identifications, $e_\pi I e_\pi$ is thus the ideal of $k[\mathfrak{S}_{r_0}] \otimes_k k[\mathfrak{S}_{r_1}]$ generated by $y_0 \otimes 1$ and $1 \otimes y_1$, where y_i is a_{i,l_i+1} in $k[\mathfrak{S}_{l_i+1}] \subset k[\mathfrak{S}_{r_i}]$ if $r_i > l_i$ and $y_i = 0$ otherwise. Further (3.7) is the tensor product of the homomorphisms

$$(3.8) \qquad\qquad k[\mathfrak{S}_{r_i}] \to \mathrm{End}_G(E_i^{\otimes r_i})$$

of k-algebras sending $\tau \in \mathfrak{S}_{r_i}$ to $E_i^{\otimes \tau}$ in $\mathrm{REP}(G, \rho)$. It will thus suffice to prove that (3.8) is surjective with kernel generated by y_i. If $i = 0$, (3.8) may be identified with the homomorphism defined by the action of $\mathfrak{S}_{r_i}$ by symmetries on the r_ith tensor power in $\mathrm{REP}(GL_{l_i})$ of the standard representation of GL_{l_i}, while if $i = 1$, the composite of the automorphism $\tau \mapsto \mathrm{sgn}(\tau)\tau$ of $k[\mathfrak{S}_{r_i}]$ with (3.8) may be so identified. The required result is thus classical (e.g. [3], Theorem 6.3). $\qquad\square$

4. DUALS

Let $\mathcal{C}$ be a k-tensor category. By a duality pairing in $\mathcal{C}$ we mean a quadruple $(L, L^\vee, \eta, \varepsilon)$ consisting of objects L and $L^\vee$ of $\mathcal{C}$ and morphisms $\eta : \mathbf{1} \to L^\vee \otimes L$, the unit, and $\varepsilon : L \otimes L^\vee \to \mathbf{1}$, the counit, satisfying triangular identities analogous for those of an adjunction ([5], p. 85). Explicitly, it is required that, modulo associativities, the composite of $L \otimes \eta$ with $\varepsilon \otimes L$ should be 1_L, and of $\eta \otimes L^\vee$ with $L^\vee \otimes \varepsilon$ should be $1_{L^\vee}$. When such an $(L, L^\vee, \eta, \varepsilon)$ exists for a given L, it is said to be a duality pairing for L, and L is said to be dualisable, and $L^\vee$ to be dual to L. We then have a dual pairing $(L^\vee, L, \tilde\eta, \tilde\varepsilon)$ for $L^\vee$, with $\tilde\eta$ and $\tilde\varepsilon$ obtained from η and ε by composing with the appropriate symmetries.

In verifying the properties of duals recalled below, it is useful to reduce to the case where $\mathcal{C}$ is strict, i.e. where all associativities of $\mathcal{C}$ are identities. This can be

done by taking (see [5], XI 3, Theorem 1) a k-linear strong symmetric monoidal functor ([5], XI 2) $\mathcal{C} \to \mathcal{C}'$ giving an equivalence to a strict k-tensor category $\mathcal{C}'$.

Suppose given duality pairings $(L, L^\vee, \eta, \varepsilon)$ for L and $(L', L'^\vee, \eta', \varepsilon')$ for L'. Then we have a tensor product duality pairing for $L \otimes L'$, with dual $L^\vee \otimes L'^\vee$, and unit and counit obtained from $\eta \otimes \eta'$ and $\varepsilon \otimes \varepsilon'$ by composing with the appropriate symmetries. Further any morphism $f : L \to L'$ has a transpose $f^\vee : L'^\vee \to L^\vee$, characterised by the condition

$$\varepsilon \circ (L \otimes f^\vee) = \varepsilon' \circ (f \otimes L'^\vee),$$

or by a similar condition using η and η'. Explicitly, $f^\vee$ is given modulo associativities by the composite of $\eta \otimes L'^\vee$ with $L^\vee \otimes f \otimes L'^\vee$ and $L^\vee \otimes \varepsilon'$. We have $(1_L)^\vee = 1_{L^\vee}$ and $(f' \circ f)^\vee = f^\vee \circ f'^\vee$, and, with the transpose of $f^\vee$ taken using the dual pairing, we have $f^{\vee\vee} = f$. In particular taking $L = L'$ shows that a duality pairing for L is unique up to unique isomorphism.

Let L be a dualisable object of $\mathcal{C}$. Then we have a k-linear map

$$\mathrm{tr}_L : \mathrm{Hom}_\mathcal{C}(N \otimes L, N' \otimes L) \xrightarrow{\sim} \mathrm{Hom}_\mathcal{C}(N, N'),$$

natural in N and N', which sends f to its contraction $\mathrm{tr}_L(f)$ with respect to L, defined as follows. Modulo associativities, $\mathrm{tr}_L(f)$ is the composite of $N \otimes \widetilde{\eta}$ with $f \otimes L^\vee$ and $N' \otimes \varepsilon$, with $L^\vee$ and ε as above and $\widetilde{\eta}$ the composite of η with the symmetry interchanging $L^\vee$ and L. It does not depend on the choice of duality pairing for L. When $N = N' = \mathbf{1}$, the contraction $\mathrm{tr}_L(f)$ is the trace $\mathrm{tr}(f)$ of the endomorphism f of L. The rank of L is defined as $\mathrm{tr}(1_L)$. Modulo associativities, $\mathrm{tr}_{L \otimes L'}$ is given by successive contraction with respect to L' and L, and tr_L commutes with $M \otimes -$. By the appropriate triangular identity for L we have

$$(4.1) \qquad\qquad g'' \circ g' = \mathrm{tr}_L((g'' \otimes g') \circ \sigma)$$

for $g' : M' \to L$ and $g'' : L \to M''$, with σ the symmetry interchanging M' and L.

Let L be a dualisable object of $\mathcal{C}$, and let τ be a permutation of $[1, r+1]$ and $f_1, f_2, \ldots, f_{r+1}$ be endomorphisms of L. Write τ' for the permutation of $[1, r]$ obtained by omitting $r+1$ from the cycle of τ containing it, and define endomorphisms c of $\mathbf{1}$ and $f'_1, f'_2, \ldots, f'_r$ of L as follows. If τ leaves $r+1$ fixed, then $c = \mathrm{tr}(f_{r+1})$ and $f'_i = f_i$ for $i \le r$. If τ sends $r+1$ to $i_0 \le r$, then $c = 1$, and f'_i for $i \le r$ is f_i when $i \ne i_0$ and $f_{i_0} \circ f_{r+1}$ when $i = i_0$. We then have

$$(4.2) \qquad \mathrm{tr}_L((f_1 \otimes f_2 \otimes \cdots \otimes f_{r+1}) \circ L^{\otimes \tau}) = c((f'_1 \otimes f'_2 \otimes \cdots \otimes f'_r) \circ L^{\otimes \tau'}).$$

To see this, reduce to the case where τ leaves all but the last two elements of $[1, r+1]$ fixed, by composing on the left and right with appropriate morphisms $L^{\otimes \tau_0} \otimes L$ with τ_0 a permutation of $[1, r]$.

Let L, L', M and M' be objects in $\mathcal{C}$, and $(L, L^\vee, \eta, \varepsilon)$ and $(L', L'^\vee, \eta', \varepsilon')$ be duality pairings for L and L'. Then we have a canonical isomorphism

$$\mathrm{Hom}_\mathcal{C}(M, M' \otimes L) \xrightarrow{\sim} \mathrm{Hom}_\mathcal{C}(M \otimes L^\vee, M')$$

which modulo associativities sends $f : M \to M' \otimes L$ to the composite of $f \otimes L^\vee$ with $M' \otimes \varepsilon$. Its inverse is defined using η. We also have a canonical isomorphism

$$\mathrm{Hom}_\mathcal{C}(L' \otimes M, M') \xrightarrow{\sim} \mathrm{Hom}_\mathcal{C}(M, L'^\vee \otimes M')$$

defined using η'. Replacing M by $L' \otimes M$ in the first of these isomorphisms and by $M \otimes L^\vee$ in the second, and using the symmetries interchanging M and L' and $L'^\vee$

and M', then gives a canonical isomorphism

$$\delta_{M,L;M',L'} : \mathrm{Hom}_{\mathcal{C}}(M \otimes L', M' \otimes L) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{C}}(M \otimes L^{\vee}, M' \otimes L'^{\vee}).$$

Modulo associativities, $\delta_{M,L;M',L'}$ sends f to the composite of $M \otimes \widetilde{\eta}' \otimes L^{\vee}$, the tensor product of f with the symmetry interchanging $L'^{\vee}$ and $L^{\vee}$, and $M' \otimes \varepsilon \otimes L'^{\vee}$, where $\widetilde{\eta}'$ is η' composed with the symmetry interchanging $L'^{\vee}$ and L'.

With the transpose taken using the chosen duality pairings for L and L', we have

(4.3) $$\delta_{M,L;M',L'}(h \otimes g) = h \otimes g^{\vee}.$$

With the duality pairing $(\mathbf{1}, \mathbf{1}, 1_{\mathbf{1}}, 1_{\mathbf{1}})$ for $\mathbf{1}$, we have

(4.4) $$\delta_{M,L;\mathbf{1},\mathbf{1}}(f) = \varepsilon \circ (f \otimes L^{\vee}).$$

With the tensor product duality pairings for $L_1 \otimes L_2$ and $L_1' \otimes L_2'$, we have

(4.5) $\sigma''' \circ (\delta_{M_1,L_1;M_1',L_1'}(f_1) \otimes \delta_{M_2,L_2;M_2',L_2'}(f_2)) \circ \sigma'' =$
$$= \delta_{M_1 \otimes M_2, L_1 \otimes L_2; M_1' \otimes M_2', L_1' \otimes L_2'}(\sigma' \circ (f_1 \otimes f_2) \circ \sigma)$$

where each of σ, σ', σ'' and σ''' is a symmetry interchanging the middle two factors in a tensor product $(- \otimes -) \otimes (- \otimes -)$.

If M' is dualisable, we have

(4.6) $\delta_{M',L';M'',L''}(f') \circ \delta_{M,L;M',L'}(f) = \delta_{M,L;M'',L''}(\mathrm{tr}_{M' \otimes L'}(\sigma_2 \circ (f' \otimes f) \circ \sigma_1))$

for σ_1 the symmetry interchanging M and M' and σ_2 the symmetry interchanging L' and L. This can be seen by showing that modulo associativities both sides of (4.6) coincide with a morphism obtained from

$$f' \otimes f \otimes L^{\vee} \otimes L''^{\vee} \otimes M'^{\vee} \otimes L'^{\vee}$$

as follows: compose on the left and right with appropriate symmetries, then on the left with the tensor product of $M'' \otimes L''^{\vee}$ and the counits for L, L' and M' and on the right with the tensor product of $M \otimes L^{\vee}$ with the units for L'', L' and M'. To show this in the case of the left hand side of (4.6), write it as a contraction with respect to $M' \otimes L'^{\vee}$ using (4.1) and contract first with respect to $L'^{\vee}$, using the triangular identity.

With the duality pairing $(\mathbf{1}, \mathbf{1}, 1_{\mathbf{1}}, 1_{\mathbf{1}})$ for $\mathbf{1}$ and the tensor product duality pairing for $L \otimes N$, we have

(4.7) $\delta_{M,N;\mathbf{1},\mathbf{1}}(g \circ f) = (\delta_{M,L;\mathbf{1},\mathbf{1}}(f) \otimes \delta_{L,N;\mathbf{1},\mathbf{1}}(g)) \circ \sigma \circ \delta_{M,N;M \otimes L, L \otimes N}(\alpha),$

with $\alpha : M \otimes (L \otimes N) \xrightarrow{\sim} (M \otimes L) \otimes N$ the associativity and σ the symmetry interchanging L and $L^{\vee}$ in the tensor product of $M \otimes L$ and $L^{\vee} \otimes N^{\vee}$. Indeed modulo associativities $\sigma \circ \delta_{M,N;M \otimes L, L \otimes N}(\alpha)$ is $1_M \otimes \eta \otimes 1_{N^{\vee}}$ by the triangular identity for N, and (4.7) then follows by the triangular identity for L.

Let $(L, L^{\vee}, \eta, \varepsilon)$ be a duality pairing for the object L of $\mathcal{C}$. Its rth tensor power $(L^{\otimes r}, (L^{\vee})^{\otimes r}, \eta_r, \varepsilon_r)$ is a duality pairing for $L^{\otimes r}$. We write

$$L^{r,s} = L^{\otimes r} \otimes (L^{\vee})^{\otimes s}.$$

Then $L^{r,0} = L^{\otimes r}$. We define a k-bilinear product $\widetilde{\otimes}$ on morphisms between the $L^{r,s}$ by requiring that the square

$$(4.8) \quad \begin{array}{ccc} L^{r_1,s_1} \otimes L^{r_2,s_2} & \xrightarrow{\ \sim\ } & L^{r_1+r_2,s_1+s_2} \\ {\scriptstyle f_1 \otimes f_2}\downarrow & & \downarrow{\scriptstyle f_1\widetilde{\otimes}f_2} \\ L^{r_1',s_1'} \otimes L^{r_2',s_2'} & \xrightarrow{\ \sim\ } & L^{r_1'+r_2',s_1'+s_2'} \end{array}$$

commute, with the top isomorphism the symmetry interchanging the two factors $(L^\vee)^{\otimes s_1}$ and $L^{\otimes r_2}$ and the bottom that interchanging $(L^\vee)^{\otimes s'_1}$ and $L^{\otimes r'_2}$. Then $\widetilde{\otimes}$ preserves composites, is associative, and we have

$$(4.9) \qquad f_2 \widetilde{\otimes} f_1 = \sigma' \circ (f_1 \widetilde{\otimes} f_2) \circ \sigma^{-1},$$

where σ interchanges the first r_1 with the last r_2 factors L and the first s_1 with the last s_2 factors $L^\vee$ of $L^{r_1+r_2,s_1+s_2}$, and similarly for σ'. We define an isomorphism

$$(4.10) \qquad \delta_{L;r,s;r',s'} : \operatorname{Hom}_{\mathcal{C}}(L^{\otimes(r+s')}, L^{\otimes(r'+s)}) \xrightarrow{\sim} \operatorname{Hom}_{\mathcal{C}}(L^{r,s}, L^{r',s'})$$

by taking $L^{\otimes r}, L^{\otimes s}, L^{\otimes r'}, L^{\otimes s'}$ for M, L, M', L' in $\delta_{M,L;M',L'}$.

It follows from (4.3) that

$$(4.11) \qquad \delta_{L;r,s;r',s'}(h \otimes g) = h \otimes g^\vee.$$

and from (4.4) that

$$(4.12) \qquad \delta_{L;r,s;0,0}(f) = \varepsilon_s \circ (f \otimes (L^\vee)^{\otimes s}).$$

By (4.5), we have

$$(4.13) \quad \delta_{L;r_1,s_1;r_1',s_1'}(f_1) \widetilde{\otimes} \delta_{L;r_2,s_2;r_2',s_2'}(f_2) =$$
$$= \delta_{L;r_1+r_2,s_1+s_2;r_1'+r_2',s_1'+s_2'}(\sigma' \circ (f_1 \otimes f_2) \circ \sigma)$$

for appropriate symmetries σ and σ'. By (4.6), we have

$$(4.14) \quad \delta_{L;r',s';r'',s''}(f') \circ \delta_{L;r,s;r',s'}(f) = \delta_{L;r,s;r'',s''}(\operatorname{tr}_{L^{\otimes(r'+s')}}(\sigma_2 \circ (f' \otimes f) \circ \sigma_1)).$$

for appropriate symmetries σ_1 and σ_2. We have

$$(4.15) \quad \delta_{L;r,t;0,0}(g \circ f) = (\delta_{L;r,s;0,0}(f) \widetilde{\otimes} \delta_{L;s,t;0,0}(g)) \circ \delta_{L;r,t;r+s,s+t}(1_{L^{\otimes(r+s+t)}}),$$

by (4.7).

Let G be a linear algebraic group over k and ρ be a central k-point of G with $\rho^2 = 1$. Let E be a finite-dimensional G-module and R be a commutative algebra in $\operatorname{REP}(G,\rho)$. Then E in $\operatorname{REP}(G,\rho)$ and E_R in the k-tensor category of (G,R)-modules are dualisable. Suppose chosen duality pairings for E and E_R. Then we have a G-module $E^{r,s}$ and a (G,R)-module $(E_R)^{r,s}$ for every r and s. We have canonical embeddings $E \to E_R$ and $E^\vee \to (E_R)^\vee$, which are compatible with the units and counits of the chosen duality pairings for E and E_R. They define a canonical embedding $E^{r,s} \to (E_R)^{r,s}$, which induces an isomorphism of (G,R)-modules $(E^{r,s})_R \xrightarrow{\sim} (E_R)^{r,s}$. Given $u : E^{r,s} \to E^{r',s'}$, we write $u_{R;r,s;r',s'}$ for the unique morphism of (G,R)-modules for which the square

$$(4.16) \quad \begin{array}{ccc} (E_R)^{r,s} & \xrightarrow{\ u_{R;r,s;r',s'}\ } & (E_R)^{r',s'} \\ \uparrow & & \uparrow \\ E^{r,s} & \xrightarrow{\ u\ } & E^{r',s'} \end{array}$$

commutes, with the vertical arrows the canonical embeddings. Then $(-)_{R;r,s;r',s'}$ preserves identities and composites, counits $E^{r,r} \to E^{0,0}$ and $(E_R)^{r,r} \to (E_R)^{0,0}$ and (with the identification $E^{r,0} = E^{\otimes r}$) commutes with the isomorphisms δ_E and δ_{E_R}. For each r and s we have an isomorphism

$$(4.17) \qquad \psi_{r,s} : \mathrm{Hom}_{G,R}((E_R)^{r,s}, R) \xrightarrow{\sim} \mathrm{Hom}_G(E^{r,s}, R),$$

given by composing with the canonical embedding $E^{r,s} \to (E_R)^{r,s}$. Then

$$(4.18) \qquad \psi_{r,s}(w' \circ u_{R;r,s;r',s'}) = \psi_{r',s'}(w') \circ u$$

for every $w' : (E_R)^{r',s'} \to R$ and $u : E^{r,s} \to E^{r',s'}$.

Suppose that G is reductive and that $R^G = k$, so that R has a unique maximal G-ideal J. Let N be a dualisable (G,R)-module and $f : R \to N$ be a morphism of (G,R)-modules which does not factor through JN. Then f has a left inverse. Indeed $f^\vee$ does not factor through J, because f is the composite of the unit for $N^\vee$ with $N \otimes_R f^\vee$. Hence $f^\vee$ is surjective, and there is an x in its source fixed by G with $f^\vee(x) = 1$. Thus $f^\vee$ has a unique right inverse g with $g(1) = x$, and $g^\vee$ is left inverse to $f = f^{\vee\vee}$.

5. Kimura objects

Let k be a field of characteristic 0 and $\mathcal{C}$ be a k-tensor category with $\mathrm{End}_\mathcal{C}(\mathbf{1}) = k$. An object L of $\mathcal{C}$ will be called positive (resp. negative) if it is dualisable and $\bigwedge^{r+1} L$ (resp. $S^{r+1}L$) is 0 for some r. An object of $\mathcal{C}$ will be called a Kimura object if it is the direct sum of a positive and a negative object of $\mathcal{C}$.

Let L be a Kimura object of $\mathcal{C}$. Then $L = L_0 \oplus L_1$ with L_0 positive and L_1 negative. Denote by l_0 (resp. l_1) the least r such that $\bigwedge^{r+1} L_0$ (resp. $S^{r+1}L_1$) is 0, and let G and E be as in (3.2) and (3.3), and ρ be the central k-point of G which acts as $(-1)^i$ on E_i. The goal of this section is to construct a commutative algebra R in $\mathrm{REP}(G,\rho)$ and an isomorphism

$$(5.1) \qquad \xi_{r,s} : \mathrm{Hom}_{G,R}((E_R)^{\otimes r}, (E_R)^{\otimes s}) \xrightarrow{\sim} \mathrm{Hom}_\mathcal{C}(L^{\otimes r}, L^{\otimes s})$$

for every r and s, such that the ξ preserve composites and symmetries and are compatible with $\otimes_R$ and $\otimes$.

Given an object M of $\mathcal{C}$, write $a_{M,0,r}$ (resp. $a_{M,1,r}$) for the image of the antisymmetrising (resp. symmetrising) idempotent of $k[\mathfrak{S}_r]$ under the k-homomorphism to $\mathrm{End}(M^{\otimes r})$ that sends τ in $\mathfrak{S}_r$ to $M^{\otimes \tau}$. If M is dualisable of rank d, then applying (4.2) with the f_j the identities shows that

$$(r+1)\,\mathrm{tr}_M(a_{M,i,r+1}) = (d - (-1)^i r)a_{M,i,r}$$

for $i = 0, 1$. If M is positive (resp. negative), it follows that d (resp. $-d$) is the least r for which $\bigwedge^{r+1} M$ (resp. $S^{r+1}M$) is 0. Thus L_i has rank $(-1)^i l_i$.

Write b for the automorphism of L that sends L_i to L_i and acts on it as $(-1)^i$. Then for every r, the group Γ_r of (3.4) acts on $L^{\otimes r}$ with the action of $(\mathbf{Z}/2)^r$ the rth tensor power of the action $i \mapsto b^i$ of $\mathbf{Z}/2$ on L, and the action of $\mathfrak{S}_r$ that given by $\tau \mapsto L^{\otimes \tau}$. Thus we obtain a homomorphism

$$\alpha_r : k[\Gamma_r] \to \mathrm{End}_\mathcal{C}(L^{\otimes r})$$

of k-algebras. If $l_i < r$, then α_i sends the element $x_{i,r}$ of (3.6) to the projection onto the direct summand $\bigwedge^{l_0+1} L_0 \otimes_k L^{\otimes(r-l_0-1)}$ when $i = 0$ and $S^{l_1+1}L_1 \otimes_k L^{\otimes(r-l_0-1)}$ when $i = 1$. Thus both $x_{0,r}$ and $x_{1,r}$ lie in the kernel of α_r. If we write β_r for (3.5),

it follows by Lemma 3.2 (ii) that the kernel of α_r contains that of β_r. Hence by Lemma 3.2 there is for each r and r' a unique k-linear map

$$\varphi_{r;r'} : \mathrm{Hom}_G(E^{\otimes r}, E^{\otimes r'}) \to \mathrm{Hom}_{\mathcal{C}}(L^{\otimes r}, L^{\otimes r'})$$

such that

$$\alpha_r = \varphi_{r;r} \circ \beta_r$$

for every r. By construction, the $\varphi_{r;r'}$ preserve symmetries, identities and composites, and they are compatible with $\otimes_k$ and $\otimes$. Applying (4.2) t times shows that for $v : E^{\otimes(r+t)} \to E^{\otimes(r'+t)}$ we have

(5.2) $$\varphi_{r;r'}(\mathrm{tr}_{E^{\otimes t}}(v)) = \mathrm{tr}_{L^{\otimes t}}(\varphi_{r+t;r'+t}(v)),$$

because $\mathrm{tr}(\rho^i) = \mathrm{tr}(b^i) = l_0 - (-1)^i l_1$.

For every r, s and r', s', we define a k-linear map $\varphi_{r,s;r's'}$ by requiring that the square

$$
\begin{array}{ccc}
\mathrm{Hom}_G(E^{r,s}, E^{r',s'}) & \xrightarrow{\varphi_{r,s;r's'}} & \mathrm{Hom}_{\mathcal{C}}(L^{r,s}, L^{r',s'}) \\
{\scriptstyle \delta_{E;r,s;r's'}} \uparrow & & \uparrow {\scriptstyle \delta_{L;r,s;r's'}} \\
\mathrm{Hom}_G(E^{\otimes(r+s')}, E^{\otimes(r'+s)}) & \xrightarrow{\varphi_{r+s';r'+s}} & \mathrm{Hom}_{\mathcal{C}}(L^{\otimes(r+s')}, L^{\otimes(r'+s)})
\end{array}
$$

commute, with the δ the isomorphisms of (4.10). Then by (4.11) the $\varphi_{r,s;r's'}$ preserve identities, and by (4.14) and (5.2) they preserve composites. By (4.13), they are compatible with the bilinear products, defined as in (4.8), $\widetilde{\otimes}_k$ on G-homomorphisms between the $E^{r,s}$ and $\widetilde{\otimes}$ on morphisms between the $L^{r,s}$. By (4.11), they send symmetries permuting the factors E or $E^\vee$ of $E^{r,s}$ to the corresponding symmetries of $L^{r,s}$.

We now define as follows a commutative algebra R in $\mathrm{REP}(G, \rho)$. Consider the small category $\mathcal{L}$ whose objects are triples (r, s, f) with r and s integers ≥ 0 and $f : L^{r,s} \to \mathbf{1}$, where a morphism from (r, s, f) to (r', s', f') in $\mathcal{L}$ is a morphism $u : E^{r,s} \to E^{r',s'}$ such that

$$f = f' \circ \varphi_{r,s;r',s'}(u).$$

Then we define R as the colimit

$$R = \mathop{\mathrm{colim}}_{(r,s,f)\in\mathcal{L}} E^{r,s}$$

in $\mathrm{REP}(G, \rho)$. Write the colimit injection at (r, s, f) as

$$i_{(r,s,f)} : E^{r,s} \to R.$$

We define the unit $\mathbf{1} \to R$ of R as $i_{(0,0,1_1)}$. We define the multiplication $R \otimes_k R \to R$ by requiring that for every $((r_1, s_1, f_1), (r_2, s_2, f_2))$ in $\mathcal{L} \times \mathcal{L}$ the square

$$
\begin{array}{ccc}
E^{r_1,s_1} \otimes_k E^{r_2,s_2} & \xrightarrow{\sim} & E^{r_1+r_2,s_1+s_2} \\
{\scriptstyle i_{(r_1,s_1,f_1)} \otimes_k i_{(r_1,s_1,f_2)}} \downarrow & & \downarrow {\scriptstyle i_{(r_1+r_2,s_1+s_2,f_1 \widetilde{\otimes} f_2)}} \\
R \otimes_k R & \longrightarrow & R
\end{array}
$$

(5.3)

should commute, where the top isomorphism is that of (4.8) with E for L. Such an $R \otimes_k R \to R$ exists and is unique because the left vertical arrows of the squares (5.3) form a colimiting cone by the fact that $\otimes_k$ preserves colimits, while their top right legs form a cone by the compatibility of the $\varphi_{r,s;r's'}$ with $\widetilde{\otimes}_k$ and $\widetilde{\otimes}$. The associativity of the multiplication can be checked by writing $R \otimes_k R \otimes_k R$ as a

colimit over $\mathcal{L} \times \mathcal{L} \times \mathcal{L}$ and using the associativity of $\widetilde{\otimes}$. The commutativity follows from (4.9) and the compatibility of the $\varphi_{r,s;r's'}$ with the symmetries.

Since G is reductive, each $\mathrm{Hom}_G(E^{r,s}, -)$ preserves colimits. Hence the

$$\mathrm{Hom}_G(E^{r,s}, i_{(r',s',f')}) : \mathrm{Hom}_G(E^{r,s}, E^{r',s'}) \to \mathrm{Hom}_G(E^{r,s}, R).$$

form a colimiting cone of k-vector spaces. Thus for every r and s there is a unique homomorphism

$$\theta_{r,s} : \mathrm{Hom}_G(E^{r,s}, R) \to \mathrm{Hom}_{\mathcal{C}}(L^{r,s}, \mathbf{1})$$

whose composite with $\mathrm{Hom}_G(E^{r,s}, i_{(r',s',f')})$ sends $u : E^{r,s} \to E^{r',s'}$ to

$$f' \circ \varphi_{r,s;r',s'}(u).$$

Further $\theta_{r,s}$ is an isomorphism, with inverse sending $f : L^{r,s} \to \mathbf{1}$ to $i_{(r,s,f)}$. Thus every $E^{r,s} \to R$ can be written uniquely in the form $i_{(r,s,f)}$. It follows that

$$(5.4) \qquad \theta_{r,s}(v' \circ u) = \theta_{r',s'}(v') \circ \varphi_{r,s;r',s'}(u)$$

for $v' : E^{r',s'} \to R$, that $\theta_{0,0}$ sends the identity $k \to R$ of R to $1_{\mathbf{1}}$, and that

$$(5.5) \qquad \theta_{r_1+r_2,s_1+s_2}(v) = \theta_{r_1,s_1}(v_1) \widetilde{\otimes} \theta_{r_2,s_2}(v_2)$$

for $v_1 : E^{r_1,s_1} \to R$ and $v_2 : E^{r_2,s_2} \to R$, where v is defined by a diagram of the form (5.3) with left arrow $v_1 \otimes_k v_2$ and right arrow v.

Composing the isomorphisms $\psi_{r,s}$ of (4.17) and $\theta_{r,s}$ gives an isomorphism

$$\widehat{\theta}_{r,s} = \theta_{r,s} \circ \psi_{r,s} : \mathrm{Hom}_{G,R}((E_R)^{r,s}, R) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{C}}(L^{r,s}, \mathbf{1}).$$

Then with $u_{R;r,s;r',s'}$ as in (4.16), we have by (4.18) and (5.4)

$$(5.6) \qquad \widehat{\theta}_{r,s}(w' \circ u_{R;r,s;r',s'}) = \widehat{\theta}_{r',s'}(w') \circ \varphi_{r,s;r',s'}(u)$$

for every $w' : (E_R)^{r',s'} \to R$ and $u : E^{r,s} \to E^{r',s'}$. Also $\widehat{\theta}_{0,0}(1_R) = 1_{\mathbf{1}}$, and

$$(5.7) \qquad \widehat{\theta}_{r_1+r_2,s_1+s_2}(w_1 \widetilde{\otimes}_R w_2) = \widehat{\theta}_{r_1,s_1}(w_1) \widetilde{\otimes} \widehat{\theta}_{r_2,s_2}(w_2)$$

for every $w_1 : (E_R)^{r_1,s_1} \to R$ and $w_2 : (E_R)^{r_2,s_2} \to R$, by (5.5).

We now define the isomorphism (5.1) by requiring that the square

$$\begin{CD}
\mathrm{Hom}_{G,R}((E_R)^{\otimes r}, (E_R)^{\otimes s}) @>\xi_{r,s}>> \mathrm{Hom}_{\mathcal{C}}(L^{\otimes r}, L^{\otimes s}) \\
@V\delta_{E_R;r,s,0,0}VV @VV\delta_{L;r,s,0,0}V \\
\mathrm{Hom}_{G,R}((E_R)^{r,s}, R) @>\widehat{\theta}_{r,s}>> \mathrm{Hom}_{\mathcal{C}}(L^{r,s}, \mathbf{1})
\end{CD}$$

commute. The ξ preserve composites by (4.15), (5.6), (5.7), and the fact that $(-)_{R;r,s;r',s'}$ preserves identities and is compatible with δ_E and δ_{E_R}. They are compatible with $\otimes_R$ and $\otimes$ by (4.13), where the relevant σ and σ' reduce to associativities, and (5.7). They are compatible with the symmetries by (5.6) with $w' = 1_R$ and u the composite of $\sigma \otimes_k (E^\vee)^{\otimes r}$ for σ a symmetry of $E^{\otimes r}$ with the counit $E^{r,r} \to k$, using (4.12) and the compatibility of $(-)_{R;r,s;r',s'}$ with symmetries, composites, and counits.

6. Kimura varieties

We denote by $\mathcal{M}_\sim(F)$ the category of ungraded $\mathbf{Q}$-linear motives over F for the equivalence relation $\sim$. It is a $\mathbf{Q}$-tensor category. There is a contravariant functor h from the category $\mathcal{V}_F$ of smooth projective varieties over F to $\mathcal{M}_\sim(F)$, which sends products in $\mathcal{V}_F$ to tensor products in $\mathcal{M}_\sim(F)$. We then have

$$(6.1) \qquad \operatorname{Hom}_{\mathcal{M}_\sim(F)}(h(X'), h(X)) = CH(X' \times_F X)_{\mathbf{Q}}/\sim,$$

and the composite $z \circ z'$ of $z' : h(X'') \to h(X')$ with $z : h(X') \to h(X)$ is given by

$$z \circ z' = (\operatorname{pr}_{13})_*((\operatorname{pr}_{12})^*(z').(\operatorname{pr}_{23})^*(z)),$$

where the projections are from $X'' \times_F X' \times_F X$. Further $h(q)$ for $q : X \to X'$ is the push forward of 1 in $CH(X)_{\mathbf{Q}}/\sim$ along $X \to X' \times_F X$ with components q and 1_X.

The images under h of the structural morphism and diagonal of X define on $h(X)$ a canonical structure of commutative algebra in $\mathcal{M}_\sim(F)$. With this structure (6.1) reduces when $X' = \operatorname{Spec}(F)$ to an equality of algebras

$$\operatorname{Hom}_{\mathcal{M}_\sim(F)}(\mathbf{1}, h(X)) = CH(X)_{\mathbf{Q}}/\sim.$$

Also $h(X)$ is canonically autodual: we have canonical duality pairing

$$(h(X), h(X), \eta_X, \varepsilon_X),$$

with both η_X and ε_X the class in $CH(X \times_F X)_{\mathbf{Q}}/\sim$ of the diagonal of X. The canonical duality pairing for $h(X \times_F X')$ is the tensor product of those for $h(X)$ and $h(X')$. The canonical duality pairings define a transpose $(-)^\vee$ for morphisms $h(X') \to h(X)$, given by pullback of cycles along the symmetry interchanging X and X'. For $q : X \to X'$ and $z \in CH(X)_{\mathbf{Q}}/\sim$ and $z' \in CH(X')_{\mathbf{Q}}/\sim$, we have

$$(6.2) \qquad\qquad\qquad q^*(z') = h(q) \circ z'$$

and

$$(6.3) \qquad\qquad\qquad q_*(z) = h(q)^\vee \circ z.$$

A *Kimura variety for* $\sim$ is a smooth projective variety X over F such that $h(X)$ is a Kimura object in $\mathcal{M}_\sim(F)$. If the motive of X in the category of *graded* motives for $\sim$ is a Kimura object, then X is a Kimura variety for $\sim$. The converse also holds, as can be seen by factoring out the tensor ideals of tensor nilpotent morphisms, but this will not be needed.

Let X be a Kimura variety for $\sim$. We may apply the construction of Section 5 with $k = \mathbf{Q}$, $\mathcal{C} = \mathcal{M}_\sim(F)$ and $L = h(X)$. For appropriate l_0 and l_1, we then have with G, E and ρ as in Section 5 a commutative algebra R in $\operatorname{REP}(G, \rho)$ and isomorphisms

$$\xi_{r,s} : \operatorname{Hom}_{G,R}((E_R)^{\otimes r}, (E_R)^{\otimes s}) \xrightarrow{\sim} \operatorname{Hom}_{\mathcal{M}_\sim(F)}(h(X)^{\otimes r}, h(X)^{\otimes s})$$

which are compatible with composites, tensor products, and symmetries.

The homomorphisms of R-modules ι and μ with respective images under $\xi_{0,1}$ and $\xi_{2,1}$ the unit and multiplication of $h(X)$ define a structure of commutative R-algebra on E_R. Also the homomorphisms η_1 and ε_1 with respective images η_X and ε_X under $\xi_{0,2}$ and $\xi_{2,0}$ are the unit and counit a duality pairing $(E_R, E_R, \eta_1, \varepsilon_1)$ for E_R. We denote by

$$((E_R)^{\otimes r}, (E_R)^{\otimes r}, \eta_r, \varepsilon_r)$$

its rth tensor power. Then $\xi_{0,2r}(\eta_r) = \eta_{X^r}$ and $\xi_{2r,0}(\varepsilon_r) = \varepsilon_{X^r}$. For any (G,R)-homomorphism f from $(E_R)^{\otimes m}$ to $(E_R)^{\otimes l}$ we have

$$\xi_{l,m}(f^\vee) = \xi_{m,l}(f)^\vee,$$

where the transpose of f is taken using duality pairings just defined. Further

$$\xi_{0,n} : \operatorname{Hom}_{G,R}(R, (E_R)^{\otimes n}) \xrightarrow{\sim} \operatorname{Hom}_{\mathcal{M}_\sim(F)}(\mathbf{1}, h(X)^{\otimes n})$$

is an isomorphism of $\mathbf{Q}$-algebras. We note that

$$R^G = \operatorname{Hom}_{G,R}(R, R) = CH(\operatorname{Spec}(F))_{\mathbf{Q}}/\!\sim\; = \mathbf{Q},$$

by the isomorphism $\xi_{0,0}$.

7. Proof of Theorem 1.1

To prove Theorem 1.1, we may suppose that Z_1 contains the classes of the equidimensional components of X, and that Z_n contains the homogeneous components of each of its elements for the grading of $CH(X^n)_{\mathbf{Q}}/\!\sim$. Denote by $\mathcal{A}$ the set of those families $C = ((C_n)^i)_{n,i\in\mathbf{N}}$ with $(C_n)^0$ a $\mathbf{Q}$-subalgebra C_n of $CH(X^n)_{\mathbf{Q}}/\!\sim$ and $((C_n)^i)_{i\in\mathbf{N}}$ a filtration of the algebra C_n, such that (a), (b) and (c) of Theorem 1.1 hold. It is to be shown that there is a C in $\mathcal{A}$ which is graded, i.e. such that $(C_n)^i$ is a graded $\mathbf{Q}$-vector subspace of $CH(X^n)_{\mathbf{Q}}/\!\sim$ for each n and i. For $\lambda \in \mathbf{Q}^*$, define an endomorphism $z \mapsto \lambda * z$ of the algebra $CH(X^n)_{\mathbf{Q}}/\!\sim$ by taking $\lambda * z = \lambda^j z$ when z is homogeneous of degree j. Then the graded subspaces of $CH(X^n)_{\mathbf{Q}}/\!\sim$ are those that are stable under each $\lambda * -$. For each C in $\mathcal{A}$ we have a $\lambda * C$ in $\mathcal{A}$ with $((\lambda * C)_n)^i$ the image under $\lambda * -$ of $(C_n)^i$. Indeed $(\lambda * C)_n$ contains Z_n by the homogeneity assumption on Z_n, and p_* sends $((\lambda * C)_l)^i$ to $((\lambda * C)_m)^i$ for p as in (a), because C_l contains the classes of the equidimensional components of each factor X of X^l by the assumption on Z_1. The C in $\mathcal{A}$ that are graded are then those fixed by each $\lambda * -$. Now if $\mathcal{A}$ is non-empty, it has a least element for the ordering of the C by inclusion of the $(C_n)^i$. Such a least element will be fixed by the $\lambda * -$, and hence graded. It will thus suffice to show that $\mathcal{A}$ is non-empty.

Let G, E, ρ, R, $\xi_{r,s}$, η_r, ε_r, ι and μ be as in Section 6. With the identification

$$(7.1) \qquad\qquad \operatorname{Hom}_{\mathcal{M}_\sim(F)}(\mathbf{1}, h(X)^{\otimes n}) = CH(X^n)_{\mathbf{Q}}/\!\sim,$$

there exists a finitely generated G-subalgebra R' of R such that if we write $\beta_{m,n}$ for the homomorphism (3.1) with $P = E$, then $(\xi_{0,n})^{-1}(Z_n)$ is contained in the image of $\beta_{0,n}$ for every n, and $\eta_1 = \beta_{0,2}(\eta_1')$, $\varepsilon_1 = \beta_{2,0}(\varepsilon_1')$, $\iota = \beta_{0,1}(\iota')$ and $\mu = \beta_{2,1}(\mu')$ for some ε_1', η_1', ι' and μ'. We then have duality pairing $(E_{R'}, E_{R'}, \eta_1', \varepsilon_1')$ for $E_{R'}$, and if its rth tensor power is

$$((E_{R'})^{\otimes r}, (E_{R'})^{\otimes r}, \eta_r', \varepsilon_r'),$$

we have $\eta_r = \beta_{0,2r}(\eta_r')$ and $\varepsilon_r = \beta_{2r,0}(\varepsilon_r')$. Further ι' and μ' define a structure of commutative (G, R')-algebra on $E_{R'}$. The $\beta_{m,l}$, and hence their composites

$$\xi_{m,l}' : \operatorname{Hom}_{G,R'}((E_{R'})^{\otimes m}, (E_{R'})^{\otimes n}) \to \operatorname{Hom}_{\mathcal{M}_\sim(F)}(h(X)^{\otimes n}, h(X)^{\otimes n})$$

with the $\xi_{m,n}$, preserve identities, composition, tensor products, and transposes defined using the η_r' and ε_r'. Further $\xi_{0,n}'$ is a homomorphism of $\mathbf{Q}$-algebras.

We have $R'^G = R^G = \mathbf{Q}$. Thus by Lemma 3.1 (i), the $\mathbf{Q}$-algebra

$$\operatorname{Hom}_{G,R'}(R', (E_{R'})^{\otimes n}) \xrightarrow{\sim} \operatorname{Hom}_G(k, (E_{R'})^{\otimes n}) = ((E_{R'})^{\otimes n})^G$$

is finite-dimensional for every n. Denote by J' the unique maximal G-ideal of R'. Then we have for each n a filtration of the (G, R')-algebra $(E_{R'})^{\otimes n}$ by the G-ideals

$$J'^r(E_{R'})^{\otimes n},$$

and hence a filtration of the $\mathbf{Q}$-algebra $\operatorname{Hom}_{G,R'}(R', (E_{R'})^{\otimes n})$ by the ideals

(7.2) $$\operatorname{Hom}_{G,R'}(R', J'^r(E_{R'})^{\otimes n}).$$

Since (7.2) is isomorphic to $(J'^r(E_{R'})^{\otimes n})^G$, it is 0 for r large, by Lemma 3.1 (ii).

We now define an element C of $\mathcal{A}$ as follows. With the identification (7.1), take for C_n the image of $\xi'_{0,n}$, and for $(C_n)^r$ the image under $\xi'_{0,n}$ of (7.2). Then (b) holds.

Let $z = \xi'_{0,n}(x)$ be an element of C_n which does not lie in $(C_n)^1$. Then x does not factor through $J'(E_{R'})^{\otimes n}$. As was seen at the end of Section 4, this implies that x has a left inverse. Hence z has a left inverse $y : h(X)^{\otimes n} \to \mathbf{1}$. Identifying y with an element of $CH(X^n)_\mathbf{Q}/\sim$, the composite $y \circ z = 1_\mathbf{1}$ is the push forward of $y.z$ along the structural morphism of X^n. Thus z is not numerically equivalent to 0. The first statement of (c) follows. The second statement of (c) follows from the fact that (7.2) is 0 for r large.

Let $p : X^l \to X^m$ be as in (a). If p is defined by $\nu : [1, m] \to [1, l]$, then

$$h(p) : h(X)^{\otimes m} \to h(X)^{\otimes l}$$

is the morphism of commutative algebras in $\mathcal{M}_\sim(F)$ defined by ν. Thus

$$h(p) = \xi'_{m,l}(f)$$

for $f : (E_{R'})^{\otimes m} \to (E_{R'})^{\otimes l}$ the morphism of commutative (G, R')-algebras defined by ν. That p^* sends C_m to C_l and respects the filtrations now follows from (6.2) and the compatibility of the $\xi'_{m,l}$ with composites. That p_* sends C_l to C_m and respects the filtrations follows from (6.3) and the compatibility of the $\xi'_{m,l}$ with composites and transposes. Thus (a) holds.

8. Proof of Theorem 1.2

Let G, E, ρ, R, $\xi_{r,s}$, η_r and ε_r be as in Section 6, and suppose that the equivalence relation $\sim$ is numerical equivalence. We show first that R is G-simple, i.e. has no G-ideals other than 0 and R. Any non-zero $z : h(X)^{\otimes m} \to \mathbf{1}$ has a right inverse y, because $z \circ y$ is the push forward of $z.y$ along the structural isomorphism of X^m. The isomorphisms ξ then show that any non-zero $(E_R)^{\otimes m} \to R$ has a right inverse, and is thus surjective. Let $J \neq 0$ be a G-ideal of R. Since G is reductive and E is a faithful representation of G, the category of finite-dimensional representations of G is the pseudo-abelian hull of its full subcategory with objects the $E^{r,s}$ ([9], 3.5). Thus for some r, s there is a non-zero homomorphism of G-modules from $E^{r,s}$ to R which factors through J. It defines by the isomorphism (4.17) a non-zero homomorphism of (G, R)-modules f from $(E_R)^{r,s}$ to R which also factors through J. Since $(E_R)^{r,s}$ is isomorphic by autoduality of E_R to $(E_R)^{r+s}$, it follows that f is surjective, so that $J = R$. Thus R has no G ideals other than 0 and R.

If R_1 is the G-submodule of R on which ρ acts as -1, then the ideal of R generated by R_1 is a G-ideal $\neq R$, because the elements of R_1 have square 0. Thus $R_1 = 0$, so that R is commutative as an algebra in $\operatorname{REP}(G)$. By a theorem of Magid ([6], Theorem 4.5), the G-simplicity of R and the fact that $R^G = \mathbf{Q}$ then imply that $\operatorname{Spec}(R)_k$ is isomorphic to G_k/H for some extension k of $\mathbf{Q}$ and closed

subgroup H of G_k. Thus R is a finitely generated $\mathbf{Q}$-algebra. Hence there exists an n such that a set of generators of R is contained in the sum of the images of the G-homomorphisms $E^{r,s} \to R$ for $r + s \leq n$. We may suppose that $n \geq 2$. We show that n satisfies the requirements of Theorem 1.2.

Denote by U_m the $\mathbf{Q}$-vector subspace of $\overline{CH}(X^m)_{\mathbf{Q}} = CH(X^m)_{\mathbf{Q}}/\sim$ generated by the elements (1.1), and by

$$U_{m,l} \subset \mathrm{Hom}_{G,R}((E_R)^{\otimes m}, (E_R)^{\otimes l})$$

the inverse image of

$$U_{m+l} \subset \mathrm{Hom}_{\mathcal{M}_\sim(F)}(h(X)^{\otimes m}, h(X)^{\otimes l}) = \overline{CH}(X^{m+l})_{\mathbf{Q}}$$

under $\xi_{m,l}$. The symmetries of $(E_R)^{\otimes m}$ lie in $U_{m,m}$, because by Proposition 2.1 the symmetries of $h(X)^{\otimes m}$ lie in U_{2m}. Similarly the composite of an element of $U_{m,m'}$ with an element of $U_{m',m''}$ lies in $U_{m,m''}$, the tensor product of an element of $U_{m,l}$ with an element of $U_{m',l'}$ lies in $U_{m+m',l+l'}$, and η_m lies in $U_{0,2m}$ and ε_m lies in $U_{2m,0}$. Also $U_{m,l}$ coincides with $\mathrm{Hom}_{G,R}((E_R)^{\otimes m}, (E_R)^{\otimes l})$ for $m + l \leq n$.

Since E_R is canonically autodual, we may identify $(E_R)^{r,s}$ with $(E_R)^{\otimes(r+s)}$. The morphism $u_{R;r,s;r',s'}$ of (4.16) may then be identified with a morphism of R-modules

$$u_{R;r,s;r',s'} : (E_R)^{\otimes(r+s)} \to (E_R)^{\otimes(r'+s')},$$

and the isomorphism $\psi_{r,s}$ of (4.17) with an isomorphism

$$\psi_{r,s} : \mathrm{Hom}_{G,R}((E_R)^{\otimes(r+s)}, R) \xrightarrow{\sim} \mathrm{Hom}_G(E^{r,s}, R).$$

Then (4.18) still holds. Also we have a commutative square

$$(8.1) \qquad \begin{array}{ccc} E^{r_1,s_1} \otimes_{\mathbf{Q}} E^{r_2,s_2} & \xrightarrow{\sim} & E^{r_1+r_2,s_1+s_2} \\ {\scriptstyle \psi_{r_1,s_1}(f_1) \otimes_{\mathbf{Q}} \psi_{r_2,s_2}(f_2)} \downarrow & & \downarrow {\scriptstyle \psi_{r_1+r_2,s_1+s_2}(f)} \\ R \otimes_{\mathbf{Q}} R & \xrightarrow{} & R \end{array}$$

where the top isomorphism is that of (4.8) with E for L, the bottom arrow is the multiplication of R, and f is the composite of the appropriate symmetry of $(E_R)^{\otimes(r_1+r_2+s_1+s_2)}$ with $f_1 \otimes_R f_2$.

By Lemma 3.2, a non zero $w : E^{r',0} \to E^{r,0}$ exists only if $r = r'$, when any such w is a composite of symmetries and tensor products of endomorphisms of E. Thus $w_{R;r',0;r,0}$ lies in $U_{r',r}$ for such a w, because $n \geq 2$. Since $(-)_{R;r,s;r',s'}$ commutes with the isomorphisms δ of (4.10), it follows that $w_{R;r',s';r,s}$ lies in $U_{r'+s',r+s}$ for any $w : E^{r',s'} \to E^{r,s}$.

To prove Theorem 1.2, write

$$W_{r,s} = \psi_{r,s}(U_{r+s,0}).$$

Consider the smallest G-submodule R' of R such that $a : E^{r,s} \to R$ factors through R' for each r, s, and a in $W_{r,s}$. By (8.1), R' is a subalgebra of R. Since every $E^{r,s} \to R$ lies in $W_{r,s}$ when $r + s \leq n$, the algebra R' contains a set of generators of R. Hence $R' = R$. Given $a : E^{r,s} \to R$, there are thus a_i in W_{r_i,s_i} for $i = 1,, 2, \ldots, t$ such that the image of a lies in the sum of the images of the a_i. By semisimplicity of $\mathrm{REP}(G,\rho)$, it follows that

$$a = a_1 \circ w_1 + a_2 \circ w_2 + \cdots + a_t \circ w_t$$

for some w_i. Hence by (4.18), a lies in $W_{r,s}$. Thus $W_{r,s} = \mathrm{Hom}_G(E^{r,s}, R)$ for every r and s. It follows that $U_{m,0} = \mathrm{Hom}_{G,R}((E_R)^{\otimes m}, R)$, and hence $U_m = \overline{CH}(X^m)_{\mathbf{Q}}$, for every m. This proves Theorem 1.2.

9. Concluding remarks

Theorem 1.1 is easily generalised to the case where instead of cycles on the powers of a single Kimura variety X for $\sim$, we consider also cycles on products of a finite number of such varieties: it suffices to take for X their disjoint union and to include in Z_1 their fundamental classes. Similarly in the condition on $X^l \to X^m$ in (a), we may consider a finite number of morphisms $X^l \to X$ additional to the projections: it suffices to include in the Z_i the classes of their graphs. Suppose for example that X is an abelian variety, and let Γ be a finitely generated subgroup of $X(k)$. Then we may consider in (a) pullback and push forward along any morphism $X^l \to X^m$ which sends the identity of $X(k)^l$ to an element of Γ^m.

More generally, we can construct a small category $\mathcal{V}$, an equivalence T from $\mathcal{V}$ to the category Kimura varieties over F for $\sim$, a filtered family $(\mathcal{V}_\lambda)_{\lambda \in \Lambda}$ of (not necessarily full) subcategories $\mathcal{V}_\lambda$ of $\mathcal{V}$ with union $\mathcal{V}$, and for each λ in Λ and V in $\mathcal{V}_\lambda$ a finite-dimensional graded $\mathbf{Q}$-subalgebra $C_\lambda(V)$ of $CH(T(V))_{\mathbf{Q}}/\sim$ and a filtration $C_\lambda(V)^\bullet$ on $C_\lambda(V)$, with the following properties.

(a) Finite products exist in the $\mathcal{V}_\lambda$, and the embeddings $\mathcal{V}_\lambda \to \mathcal{V}$ preserve them.
(b) We have $C_\lambda(V)^r \subset C_{\lambda'}(V)^r$ for $\lambda \leq \lambda'$ and V in $\mathcal{V}_\lambda$, and $CH(T(V))_{\mathbf{Q}}/\sim$ for V in $\mathcal{V}_\lambda$ is the union of the $C_{\lambda'}(V)$ for $\lambda' \geq \lambda$.
(c) $T(f)^*$ sends $C_\lambda(V')$ into $C_\lambda(V)$ and $T(f)_*$ sends $C_\lambda(V)$ into $C_\lambda(V')$ for $f : V \to V'$ in $\mathcal{V}_\lambda$, and $T(f)^*$ and $T(f)_*$ preserve the filtrations.
(d) For V in $\mathcal{V}_\lambda$, the projection from $C_\lambda(V)$ to $\overline{CH}(T(V))_{\mathbf{Q}}$ is surjective with kernel $C_\lambda(V)^1$, and $C_\lambda(V)^r$ is 0 for r large.

By applying the usual construction for motives (say ungraded) to $\mathcal{V}$ and the $CH(T(V))_{\mathbf{Q}}/\sim$ we obtain a $\mathbf{Q}$-tensor category $\mathcal{M}$ and a cohomology functor from $\mathcal{V}$ to $\mathcal{M}$, and T defines a fully faithful functor from $\mathcal{M}$ to $\mathcal{M}_\sim(F)$. Similarly we obtain from $\mathcal{V}_\lambda$ and the $C_\lambda(V)$ a (not necessarily full) $\mathbf{Q}$-tensor subcategory $\mathcal{M}_\lambda$ of $\mathcal{M}$. Then each $\mathcal{M}_\lambda$ has finite-dimensional hom-spaces, and $\mathcal{M}$ is the filtered union of the $\mathcal{M}_\lambda$. A question involving a finite number of Kimura varieties, a finite number of morphisms between their products, and a finite number of morphisms between the motives of such products, thus reduces to a question in some $\mathcal{V}_\lambda$ and $\mathcal{M}_\lambda$. By (d), the projection from $\mathcal{M}_\lambda$ to the quotient of $\mathcal{M}_\sim(F)$ by its unique maximal tensor ideal is full, with kernel the unique maximal tensor ideal $\mathcal{J}_\lambda$ of $\mathcal{M}_\lambda$. Further $\mathcal{M}_\lambda$ is the limit of the $\mathcal{M}_\lambda/(\mathcal{J}_\lambda)^r$. Thus we can argue by lifting successively from the semisimple abelian category $\mathcal{M}_\lambda/\mathcal{J}_\lambda$ to the $\mathcal{M}_\lambda/(\mathcal{J}_\lambda)^r$.

Theorems 1.1 and 1.2 extend easily to the case where the base field F is replaced by a non-empty connected smooth quasi-projective scheme S over F. For the category of ungraded motives over S we then have $\mathrm{End}(\mathbf{1}) = CH(S)_{\mathbf{Q}}/\sim$, which is a local $\mathbf{Q}$-algebra with residue field $\mathbf{Q}$ and nilpotent maximal ideal. All the arguments carry over to this case, provided that Lemma 3.1 is proved in the more general form where the hypothesis "$R^G = k$" is replaced by "R^G is a local k-algebra with residue field k".

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